THE INTERACTION BETWEEN GENERAL REASONING PROCESSES AND ACHIEVEMENT IN ALGEBRA AND NOVEL PROBLEM SOLVING

LYN ENGLISH
Centre for Mathematics and Science Education
Queensland University of Technology

ELIZABETH WARREN
Australian Catholic University
McAuley Campus

Learning mathematics involves both its product (body of knowledge) and processes (ways of knowing). The mathematical reasoning processes enable the products to be developed, applied, and communicated. The role of reasoning skill per se in the learning of mathematics has received little attention, apart from studies addressing spatial ability (e.g. Tarte, 1990). Yet the importance of such processes in mathematical learning has often been acknowledged (Australian Education Council, 1990; Cockcroft Report, 1982). A vast number of research projects have investigated novel problem solving (Lester & Kroll, 1990; Schoenfeld, 1992) and the algebraic domain (Booth,1989; Kieran,1992; Kuchemann,1981; McGregor, 1991; Quinlin, 1992). Yet there seems to be a paucity of research examining the link between reasoning processes and mathematical performance in these domains. It is generally acknowledged that these processes play a role in mathematics learning (e.g. Resnik, 1987) but the exact nature of this role remains unclear. This paper will report on research which begins to explore relationships and interactions between students' general reasoning processes and their competence in solving algebraic and novel problems.

The importance of fostering student's general reasoning processes in all areas of the curriculum has been widely documented (e.g., Beyer, 1987; Halpern, 1992; Lesgold, 1988; Paul, 1990; Peterson, 1989). It has been argued that these processes enable an individual to learn more mathematics, to apply mathematics to other disciplines, and to solve mathematical problems throughout life (Fennema & Peterson, 1985). Yet there has been a paucity of research on the role of these reasoning processes in the learning of mathematics, particularly in the core content areas. There also appears little consensus on the nature of these processes and on those associated with particular mathematical learnings (Champagne, 1992). Some researchers even argue that domain-general processes are of little value in mathematical learning (e.g. Sweller, 1989, 1990). Others contend that mathematical competence requires both comprehensive knowledge structures and general reasoning processes (e.g. Alexander, 1991; Champagne, 1992; English, 1992). We adopt the latter view in our present research.

We report here on the first stage of a study which is investigating reasoning processes students apply in algebraic learning, with a particular focus on those used in interpreting, and translating between, symbolic and visual representations. The initial phase of the study was exploratory in nature and attempted to identify any relationships between students' general reasoning processes (namely spatial, logical, analogical, and patterning), their novel problem-solving skills (in particular, their preferences for a visual or symbolic approach to solution), and their understanding of pre-algebraic and early algebraic ideas (namely, generalising from patterns and tables of data, and understanding the variable construct). We considered this to be an important line of research, given that modern approaches to algebraic instruction call upon student's general reasoning and problem-solving skills. For example, students are required to perceive and generate patterns, to develop pattern "rules", and to apply these rules to new problems (Booth, 1989; Mason, Graham, Pimm, & Gower, 1985; NCTM, 1989; Pegg & Redden, 1990). We know little however, about the extent to which students' general reasoning and problem-solving processes influence their ability to understand early algebraic ideas. For example, to what extent do students' skills in manipulating number and shape patterns govern their ability to generalise from a table of numbers?

Most researchers have tended not to investigate the role of particular reasoning processes in children's algebraic learning, rather, they have looked at how students' specific knowledge, especially that of the novice, influences the nature of the processes they use (Kieran, 1989). We need to consider the other side of the coin, that is how students' existing reasoning processes determine the nature and extent of the algebraic knowledge they acquire. It seems that different modes of algebraic representation involve developing an array of powerful reasoning and problem solving processes. In particular, these appear to be spatial thinking (including a facility with mental manipulation of shapes and the ability to change perceptual perspective), logical and analogical reasoning, classifying and hypothesising, and an ability to perceive patterns and generalise from these (Lipman, 1985). A preference for a visual or symbolic approach to solution is also felt to play a role (Presmeg, 1986,1992). This study begins to probe the role of these reasoning processes in the algebraic and problem solving domain. In this paper, only the significant findings relating specifically to the algebraic domain are reported.

THE STUDY

Methodology

Since our initial study investigated students' general reasoning and problem-solving processes and their impact on students' ability to interpret early algebraic ideas, we considered a correlational research design to be the most appropriate to use(Isaac & Michael,1985).

Seven written tests were developed. These consisted of five reasoning tests, one algebra test and one problem-solving test. Each test measured different processes and understandings.

Logical reasoning, analogical reasoning, and pattern generalisation were each measured by a separate test, each comprising ten items. Since spatial reasoning consists of two distinct component, spatial visualisation (the ability to mentally manipulate, twist or invert a visual stimuli) and spatial orientation (the ability to change ones perceptual perspective when viewing an object) Tarte (1990), two tests were used to measure this reasoning process. All items for the five reasoning tests were adapted from a wide range of commercially available materials (e.g. Kit of Factor-referenced cognitive tests).

To test children's algebraic understanding, a number of different item types were developed. The items tested children's ability to; complete patterns and tables and generalise from this data to an algebraic expression, understand the variable concept in a variety of contexts, and apply algebraic concepts.Questions were drawn and adapted from a range of sources (e.g.Booth & Blane, 1992; Kucheman, 1981; Quinlin, 1992).

The problem-solving test was developed from a number of well established sources (e.g. Burton, 1984; Krutetskii, 1976; Moses, 1982; Suwarsono, 1982) and was designed to measure students' novel problem-solving skills, particularly their preference for a visual or symbolic approach to solution. The problems demanded a minimum application of mathematical knowledge but relied heavily on students' general reasoning processes. They could be solved by a variety of strategies, including both visual and non-visual means.

Nature of the Sample

Since the study was concerned with children's understanding of beginning algebra concepts, children were chosen from Year 7 and Year 8 (mean age 12 years and 9 months) as these are the two years when algebra is introduced into the curriculum. The sample comprised of one 116 children drawn from one independent boys' school and one independent coeducational school in the Brisbane metropolitan area. Each student completed all seven tests.

RESULTS

Reliability of the items chosen for the tests

The reliability of the test items was determined by calculating the Cronbach alphas. Table 1 presents a summary of the results.

Table 1
Reliability analysis scale for the tests of general reasoning, algebra & problem solving

Test (no of items)	Cronbach Alpha	Test (no of items)	Cronbach Alpha
Logical (10)	0.6402	Spatial visual (10)	0.5490
Analogical (10)	0.6628	Spatial orient (20)	0.6192
Patterning (10)	0.6904	Algebra (22)	0.5721
		Problem solving (15)	0.7221

Considering the number of items in each test and the size of the sample these reliability coefficients can be regarded as more than adequate.

Interactions between the variables

As each test measured different reasoning processes and understandings, the seven tests addressed seven different sets of performance variables. The aggregated results from each test were used to ascertain relationships between these variables. A Pearson correlational analysis was used to identify any interactions. As it was hypothesised that all the reasoning processes would contribute to the ability to solve algebraic and novel problems, one-tailed tests were used. A summary of these results is presented in Table 2.

Table 2
Correlations between the seven variables

	logical	analog	pattern	spatial visual	spatial orient	algebra	problem solving
logical	1.0000	.5330**	.3125**	.2499*	.1250*	.5483**	.3888**
analog		1.0000	.3362**	.3057**	.1459	.4148**	.2902**
pattern			1.0000	.2415*	.1409	.2050	.1694
spatial visual				1.0000	.2357*	.2283*	.3231**
spatial orient					1.0000	.0349	.0879
algebra						1.0000	.3580**
problem solving							1.0000

**$p \leq .001$ *$p \leq .01$

Of interest to this study is the significant correlation between algebra and problem-solving (significant at the .001 level). On the whole children who were successful at algebra were also successful at problem solving. The nature of the relationship between these variables needs further investigation. The reasoning processes that contributed most significantly to success in the algebraic and problem solving domain were logical reasoning, analogical reasoning, and spatial visualisation. Both patterning and spatial orientation did not correlate significantly with either domain. This contradicts the findings of Tarte (1990), who reported that spatial orientation was significantly related to the ability to solve novel problems.

The algebra test was broken into four distinct components: generalising from visual patterns, generalising from a table of numbers, understanding the variable concept, and the application of algebraic concepts. Given the emphasis on students' ability to generalise from patterns and table of data in their early algebraic learning, we considered it important to investigate the extent to which these skills relate to student's algebraic understanding. Correlations were carried out to identify firstly, any interactions between these components and secondly, any interactions between these components and the five reasoning processes (logical, analogical, patterning, spatial visualisation, and spatial orientation). Table 3 summarises the results of the first analysis.

Table 3
Correlation between generalising the components of the algebra test

	generalising patterns	generalising tables	variable concept	application
generalising patterns	1.0000	.2741*	.1987	.3033**
generalising tables		1.0000	.3622**	.4510**
variable concept			1.0000	.3444**
application				1.0000

$^{**}p \leq .001$ $^{*}p \leq .01$

The ability to generalise form pattens correlated significantly with the ability to generalise from tables of data. This is not unexpected given that these two processes comprise a number of similarities. The correlation between the ability to generalise from a table of values and understand the variable concept was also significant (at the .001 level), suggesting that students who could generalise from a table of values also had some understanding of the variable concept. By contrast, the correlation between the ability to generalise from patterns and an understanding of the variable concept was not significant. This has implications for initial algebraic learning, given that generalising from patterns is seen as a viable alternative for introducing the variable concept (Booth, 1989; Mason et al,1985). Being able to generalise from patterns and tables of values, and understand variables, seemed to contribute significantly to the application of algebraic concepts and processes.

Various reasoning processes seemed to correlate with the four components of the algebra test. Table 4 summarises the interactions that occurred.

Table 4
Correlations between the components of the algebra test & the five reasoning processes

	generalising patterns	generalising tables	variable concept	application
logical	.2691*	.4308**	.4125**	.4371**
analogical	.2155	.3309**	.3118**	.3149**
pattern	.1840	.1136	.1121	.1843
spatial visual	.1649*	.2325**	.0624	.1363
spatial orient	.1069	.0570	-.1316	.0120

$^{**}p \leq .001$ $^{*}p \leq .01$

The results highlight the significant contribution of analogical reasoning and logical reasoning to the algebraic domain. Both are significantly correlated with understanding the variable concept, generalising from a table of values, and applying algebraic concepts and processes. The results also show a significant relationship between spatial visualisation and the ability to generalise from a table of values. By contrast, generalising from patterns is only correlated significantly with logical reasoning. Even the general patterning reasoning process has little bearing on this component of algebra.

DISCUSSION

This initial research raises a number of issues regarding the teaching of algebra. Firstly, logical reasoning, analogical reasoning, and spatial visualisation seem to have some bearing on success in the algebraic domain. Yet little opportunity seems to exist in our present curriculum for the development of these reasoning processes.

Secondly, a recent approach to teaching algebra to the beginning student uses patterning from which algebraic expressions are generated (Bennett, 1988). This approach entails introducing algebra by looking at patterns, describing the pattern, and "short handing" these descriptions into algebra (Pegg & Redden, 1990). The aim of this approach is to foster an understanding of the variable concept by seeing how algebra not only emerges naturally from these explorations but also serves as a language for expressing generality (Booth, 1989). The lack of significant correlation between the ability to generalise from a pattern and understand the variable concept questions the validity of using this approach for the introduction of the variable concept. It appears, that for the students, understanding the variable concept does not seem to be a natural progression from generalising from a pattern. Yet generalising from tables is significantly correlated with understanding the variable concept. This perhaps is a more feasible means of introducing the variable concept. Once understood, this concept can be subsequently used in drawing generalisations from patterns

Thirdly, the correlation between the general patterning process and the ability to generalise from patterns in algebra was not significant. This seems to indicate that perceiving patterns in general does not automatically lead to seeing patterns in algebra and generalising from them. It appears that we need to teach these skills within the algebraic context. This highlights the fact that children need specific instruction in not only recognising the patterns in algebra but also in drawing generalisations from them.

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